The Complete Solution of Extended Kelvin-Voigt Model

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Abstract

The great usefulness of uniaxial visco-elastic models, especially in highway engineering pavement theory, composites and other civil engineering disciplines were the reason for undertaking the trial to find a complete solution for the generalization of Kelvin-Voigt body. Here the elements of higher rank than velocities of strain and stress are considered. Carson's transformation simultaneously with residuum theorem is used for deriving solutions. The introduced procedure can be also used for more complicated differential or integral forms of constitutive equations, as well as for non homogenous initial conditions. Finally, a simple task, where the initial form of stress-strain relation limited to the case of Burgers body is examined.

Keywords: ***rheology, visco-elasticity, models***

1. Introduction

The origins of this treatise come from [1-6] papers. The list of scientific and application works concerning elementary visco-elastic models probably include more than hundred titles, due to that we limit the bibliography only to Reiner [7] and Nowacki [8] monographs in which the authors made a survey of rheological problems and models. It is important to note that Reiner made a full survey of rheological models in relation to physical rules, while Nowacki showed solving methods of rheological problems by means of Laplace transform and generalized functions.

The recalled monographs are rather old, but their contents has being repeated in many contemporary papers and can be treated as constant foundation. On the other hand, taking into account present-day paper [9], we can see the advancement of formulated earlier, in [7], problems.

In our approach we focus on benefits, which come from mathematical formalism, i.e. from admissible solutions forms for assumed constitutive equation. Although the model was used in many works, the below results have been not noticed earlier and only due to that it seems to be interesting to be presented.

On the basis of Hohenemser and Prager [7] postulate let us assume a general linear body, which is linear in Boltzmann superposition [10] sense. We confine our analyses to linear visco-elasticity, excluding inner constrains of St. Venant's type, typical for Schwedoff model [11]. Only the differential form of constitutive relation is considered. For example, for several elementary models we have:

- $\dot{\sigma}a_1 + \sigma a_0 = \dot{\varepsilon}b_1$ - Maxwell, (1.1)

- $\sigma a_0 = \varepsilon b_0 + \dot{\varepsilon}b_1$ - Kelvina-Voigt, (1.2)

- $\ddot{\sigma}a_2 + \dot{\sigma}a_1 + \sigma a_0 = \dot{\varepsilon}b_1 + \ddot{\varepsilon}b_2$ - Burgers, (1.3)

- $\dot{\sigma}a_1 + \sigma a_0 = \varepsilon b_0 + \dot{\varepsilon}b_1$ - Zener; (1.4)

where a_0, a_1, a_2, b_0, b_1, b_2 – are visco-elastic material constants,
σ, ε - are tensors of stress and strain, reduced for analyzed one dimensional problem,
number of dots over characters means the rank of time derivative.

Additionally we assume that initial conditions are homogenous, i.e. at time moment $t_0 = 0$ we adopt

$$\sigma(0) = 0, \qquad \dot{\sigma}(0) = 0, \qquad \varepsilon(0) = 0, \qquad \dot{\varepsilon}(0) = 0. \tag{2}$$

In further part of the work, during the derivation of particular formulae, there is t_0, but all the time having in the mind its value.

In mathematical sense (1.1-4) is a cutting of the following formula

$$\sum_{m=1,2,\ldots}^{\infty} a_m \frac{\partial^{(m)}\sigma}{\partial t^{(m)}} = \sum_{n=1,2,\ldots}^{\infty} b_n \frac{\partial^{(n)}\varepsilon}{\partial t^{(n)}}. \qquad (3)$$

Let us now generalize, in a simple way, models (1.1-4) taking into account (3) with m=0, 1, 2 and n=0, 1, 2 i.e. –

$$\sum_{m=0,1,\ldots}^{2} a_m \frac{\partial^{(m)}\sigma}{\partial t^{(m)}} = \sum_{n=0,1,\ldots}^{2} b_n \frac{\partial^{(n)}\varepsilon}{\partial t^{(n)}}, \qquad \text{furthermore} \qquad a_0 = 1. \qquad (4)$$

For recognizing the properties of created constructive relation (4), the Carson transform is applied. Let us now recall its definition

$$C[f(t)] = p \int_{t=0}^{t\to\infty} f(t)\exp(-p\,t)dt = \tilde{f}(p). \qquad (5)$$

2. The solution of the problem

Operating with (5) onto (4) we arrive at

$$\tilde{\varepsilon} = \frac{\tilde{\sigma}}{b_2}\frac{a_2p^2 + pa_1 + 1}{p_2 + p\beta_1 + \beta_0} = \frac{\tilde{\sigma}}{b_2}\frac{a_2p^2 + pa_1 + 1}{(p-p_1)(p-p_2)}, \qquad (6)$$

where

$$\beta_1 = \frac{b_1}{b_2},\ \beta_2 = \frac{b_0}{b_2}; \qquad (6.1)$$

p_1, p_2 - are the roots of denominator in (6) calculated by the formula

$$p_{1,2} = \frac{1}{2}\left(-\beta_0 \pm \sqrt{\Delta}\right),\ \ \Delta = (\beta_1)^2 - 4\beta_0. \qquad (7)$$

We have to analyze the set of following cases –

(I) $\Delta > 0 \quad \to \quad p_1 \neq p_2 \neq 0 \in R$,

(II) $\Delta < 0 \quad \to \quad p_1 = \overline{p}_2 \in C, \quad p_{1,2} = \alpha_0 \pm i\alpha_1$,

(III) $\Delta = 0 \quad \to \quad p_{1,2} = -\dfrac{\beta_0}{2} \neq 0$,

(IV) $\Delta = 0 \quad \to \quad p_{1,2} = 0$.

Looking for original $\varepsilon(t) = C^{-1}[\tilde{\varepsilon}]$ we modify the relation (6) to the appropriate form, adequate to apply faltung theorem –

$$\tilde{\varepsilon} b_2 = \frac{\tilde{\sigma}}{p}\left(p \frac{L(p)}{(p-p_1)(p-p_2)} \right), \qquad (8)$$

where we denote

$$L(p) = p^2 a_2 + p a_1 + 1 . \qquad (8.1)$$

By virtue of Carson transformation we have

$$C[\dot{f}] = -p f(0) + p \tilde{f} \to p \tilde{f} , \qquad (9)$$

which is valid in the case of homogenous initial conditions. Denoting

$$C[\dot{f}(t)] = p \frac{L(p)}{(p-p_1)(p-p_2)} \text{ and} \qquad (10)$$

$$\tilde{f} = \frac{L(p)}{(p-p_1)(p-p_2)} \qquad (11)$$

we can directly use the faltung form

$$\varepsilon b_2 = \int_{\tau=0}^{\tau=t} \sigma(\tau) \dot{f}(t-\tau) d\tau = \int_{\tau=0}^{\tau=t} \sigma(t-\tau) \dot{f}(\tau) d\tau , \qquad (12)$$

when the load function $\sigma(t)$ is defined.

To find the original $f(t) C^{-1}[\tilde{f}]$ the method based on residuum theorem connected with Jordan's lemma is adopted, its essence consists in using the formula

$$C^{-1}[\tilde{f}] = \sum \operatorname{Res}\left(\frac{\exp(pt)}{p} \frac{N(p)}{D(p)} \right), \qquad (13)$$

where $N(p)$ and $D(p)$ mean respectively – numerator and denominator of rational expression.

The results obtained below are illustrated by using the load function concerning constant stress $\sigma_0 \neq 0$ in time interval $t \in \langle t_0 ; t_1 \rangle$ and entirely unloading in the time period $t \in (t_1 ; \infty\rangle$ as follows –

$$\sigma = \sigma_0 [H(t-t_0) - H(t-t_1)], \qquad (14.1)$$

while $H(t)$ is Heavisidea's step function. Excluding infinitesimal time interval surrounding t_0 and t_1 - time moments the stress σ has constant value and this implies

$a_2 = a_1 = 0$ in relation (4). (14.2)

3. Solutions in particular variants

3.1. Ad. (I)

Two roots of denominator in (6) are real and non zero. The values of these roots are singular points for relation (6), as well as for (8). Using (11) and (13) we arrive at

$$f_{(I)} = \sum \mathrm{Res}\left(\frac{\exp(pt)}{p}\frac{L(p)}{(p-p_1)(p-p_2)}\right). \tag{15}$$

It is seen from (15), that we have to consider an additional singular point $p_0 = 0$. As a consequence we obtain following residua

$$p_0 \to \left[\exp(pt)\frac{L(p)}{(p-p_1)(p-p_2)}\right]_{p=0}, \tag{16.1}$$

$$p_1 \to \left[\exp\left(\frac{pt}{p}\right)\frac{L(p)}{(p-p_2)}\right]_{p=p_1}, \tag{16.2}$$

$$p_2 \to \left[\exp\left(\frac{pt}{p}\right)\frac{L(p)}{(p-p_1)}\right]_{p=p_2}. \tag{16.3}$$

The sought for function and its time derivative are read

$$f_{(I)} = \frac{1}{\beta_0} + \frac{1}{p_1 - p_2}\left[\frac{\exp(p_1 t)L(p_1)}{p_1} - \frac{\exp(p_2 t)L(p_2)}{p_2}\right], \tag{17}$$

$$\dot{f}_{(I)} = \frac{1}{p_1 - p_2}\left[\exp(p_1 t)L(p_1) - \exp(p_2 t)L(p_2)\right]. \tag{18}$$

By virtue of (12) and assumption (14.1) the strain process is described by functions –

$$\begin{aligned} b_2\varepsilon_{(I)} &= \frac{\sigma_0}{p_2 - p_1}\left\{\frac{L(p_1)}{p_1}\left[1-\exp(p_1(t-t_0))\right] - \frac{L(p_2)}{p_2}\left[1-\exp(p_2(t-t_0))\right]\right\} \to \\ &\underset{(14.2)}{\to} \frac{\sigma_0}{p_2 - p_1}\left\{\frac{1-\exp(p_1(t-t_0))}{p_1} - \frac{1-\exp(p_2(t-t_0))}{p_2}\right\} \end{aligned} \tag{19.1}$$

when $t_0 \le t \le t_1$ and

$$\begin{aligned} b_2\varepsilon_{(I)} &= \frac{\sigma_0}{p_2 - p_1}\left\{\begin{aligned} &\frac{L(p_1)}{p_1}\exp(p_1 t)\left[\exp(-p_1 t_1) - \exp(-p_1 t_0)\right] + \\ &\quad - \frac{L(p_2)}{p_2}\exp(p_2 t)\left[\exp(-p_2 t_1) - \exp(-p_2 t_0)\right] \end{aligned}\right\} \to \\ &\underset{(14.2)}{\to} \frac{\sigma_0}{p_2 - p_1}\left\{\begin{aligned} &\frac{\exp(p_1 t)}{p_1}\left[\exp(-p_1 t_1) - \exp(-p_1 t_0)\right] + \\ &\quad - \frac{\exp(p_2 t)}{p_2}\left[\exp(-p_2 t_1) - \exp(-p_2 t_0)\right] \end{aligned}\right\} \end{aligned} \tag{19.2}$$

if $t > t_1$.

3.2. Ad. (II)

In this case the roots of denominator of (6) are conjugated complex, they could be presented in an alternative algebraic or exponential form

$$p_1 = \overline{p}_2 = \alpha_0 \pm i\alpha_1 = \exp(Y_0 \pm i\,Y_1) = \rho_0 \exp(\pm i\,Y_1); \qquad i = \sqrt{-1}. \qquad (20.1)$$

Together with $p_0 = 0$ the roots of (13) form the set of singular points necessary to find the original $f_{(II)}$. Appropriately for: p_0, p_1 and p_2 the residua are -

$$p_0 = 0 \;\rightarrow\; \frac{1}{\beta_0}, \qquad (20.2)$$

$$p_1 = \alpha_0 + i\alpha_1 \;\rightarrow\; \exp(\alpha_0 t)\frac{\exp(i\alpha_1 t)}{2i\alpha_1}\frac{L(\alpha_0 + i\alpha_1)}{\alpha_0 + i\alpha_1}, \qquad (20.3)$$

$$p_2 = \alpha_0 - i\alpha_1 \;\rightarrow\; \exp(\alpha_0 t)\frac{\exp(-i\alpha_1 t)}{-2i\alpha_1}\frac{L(\alpha_0 - i\alpha_1)}{\alpha_0 - i\alpha_1}. \qquad (20.4)$$

Applying exponential form we can write

$$\frac{L(\alpha_0 \pm i\,\alpha_1)}{\alpha_0 \pm i\,\alpha_1} \underset{(14.2)}{\rightarrow} \frac{1}{\alpha_0 \pm i\,\alpha_1} = \exp(A_0 \pm i\,A_1). \qquad (20.5)$$

Using Euler’s formulae, (20.4) and summing (20.1-20.3) we find the sought for function

$$f_{(II)} = \frac{1}{\beta_0} + \frac{\exp(\alpha_0 t + A_0)}{i\alpha_1}\mathrm{sh}[i(\alpha_1 t + A_1)] = \frac{1}{\beta_0} + \frac{\exp(\alpha_0 t + A_0)}{\alpha_1}\sin(\alpha_1 t + A_1) \quad (21)$$

and their time derivative

$$\dot{f}_{(II)} = \exp(\alpha_0 t + A_0)\left[\frac{\alpha_0}{\alpha_1}\sin(\alpha_1 t + A_1) + \cos(\alpha_1 t + A_1)\right]. \qquad (22)$$

The load function (14.1) yield the strain process as follows

$$b_2\varepsilon_{(II)} = \sigma_0 \exp(B_0)$$

$$\left\{\exp[\alpha_0(t - t_0)]\left[\frac{\alpha_0}{\alpha_1}\sin(\alpha_1(t - t_0) + B_1) + \cos(\alpha_1(t - t_0) + B_1)\right] + \right. \qquad (23.1)$$

$$\left. -\left[\frac{\alpha_0}{\alpha_1}\sin(B_1) + \cos(B_1)\right]\right\}$$

for $t_0 \le t \le t_1$ and

$$b_2\varepsilon_{(II)} = \sigma_0 \exp[\alpha_0 t + B_0]$$

$$\left\{\exp[-\alpha_0 t_1]\left[\frac{\alpha_0}{\alpha_1}\sin(\alpha_1(t-t_1)+B_1)+\cos(\alpha_1(t-t_1)+B_1)\right]+\right. \qquad (23.2)$$

$$\left. -\exp[-\alpha_0 t_0]\left[\frac{\alpha_0}{\alpha_1}\sin(\alpha_1(t-t_0)+B_1)!!!+\cos(\alpha_1(t-t_0)+B_1)\right]\right\}$$

while $t > t_1$ and after simplifying denotation

$$\frac{\exp(A_0 \pm i A_1)}{\alpha_0 \pm i\alpha_1} \underset{(14.2)}{\rightarrow} \frac{1}{(\alpha_0 \pm i\alpha_1)^2} = \exp(B_0 \pm i B_1). \qquad (23.3)$$

3.3. Ad. (III)

In this variant we have double real non null root $p_1 = p_2 = -\frac{\beta_0}{2}$ which, together with $p_0 = 0$, are also singularity points for (13). The residuum for p_0 we obtain, as previously,

$$p_0 = 0 \rightarrow \frac{1}{\beta_0}. \qquad (24.1)$$

To find residuum for a double root we use the following rule

$$p_1 = p_2 = -\frac{\beta_0}{2} \rightarrow \left\{\frac{d}{dp}\left[\frac{\exp(pt)}{p}L(p)\right]\right\}_{p=p_1} = ,$$
$$= \frac{\exp(p_1 t)}{(p_1)^2}\left[L(p_1)(t-1)+p_1\dot{L}(p_1)\right] \qquad (24.2)$$

The sought for $f_{(III)}$- function we read

$$f_{(III)} = \frac{1}{\beta_0} + 4\frac{\exp\left(-\frac{\beta_0 t}{2}\right)}{(\beta_0)^2}\left[L\left(-\frac{\beta_0}{2}\right)(t-1)-\frac{\beta_0}{2}\dot{L}\left(-\frac{\beta_0}{2}\right)\right] \qquad (25)$$

and its time derivative

$$\dot{f}_{(III)} = 4\frac{\exp\left(-\frac{\beta_0 t}{2}\right)}{(\beta_0)^2}(\vartheta_1 t + \vartheta_0), \qquad (26)$$

when we denote

$$\vartheta_1 = -\frac{\beta_0}{2}L\left(-\frac{\beta_0}{2}\right) \underset{(14.2)}{\rightarrow} -\frac{\beta_0}{2}, \qquad (26.1)$$

$$\vartheta_0 = \left(\frac{\beta_0}{2}\right)^2 \dot{L}\left(-\frac{\beta_0}{2}\right) + L\left(-\frac{\beta_0}{2}\right)\left(\frac{\beta_0}{2}+1\right) \underset{(14.2)}{\rightarrow} \frac{\beta_0}{2}+1. \tag{26.2}$$

The strain process for the loading function (14.1-2) and (26) implies the following result

$$b_2\varepsilon_{(III)} = \frac{4\sigma_0}{(\beta_0)^2}\left\{-\frac{2}{\beta_0} + \exp\left(-\frac{\beta_0(t-t_0)}{2}\right)\left[\frac{2}{\beta_0} + (t-t_0)\right]\right\} \tag{27.1}$$

when $t_0 \le t \le t_1$ and

$$\begin{gathered} b_2\varepsilon_{(III)} = \frac{4\sigma_0}{(\beta_0)^2}\exp\left(-\frac{\beta_0 t}{2}\right) \\ \left\{\exp\left(\frac{\beta_0 t_0}{2}\right)\left[\frac{2}{\beta_0} + (t-t_0)\right] - \exp\left(\frac{\beta_0 t_1}{2}\right)\left[\frac{2}{\beta_0} + (t-t_1)\right]\right\}. \end{gathered} \tag{27.2}$$

3.4. Ad. (IV)

Similar as above we have double real, but in this case, null root $p_1 = p_2 = 0$. Taking into account $p_0 = 0$ we have a triple singularity point, this implies the residuum value as

$$\begin{gathered} p_0 = p_1 = p_2 = 0 \rightarrow \left\{\frac{1}{2!}\frac{d^2}{dp^2}\left[\exp(pt)L(p)\right]\right\}_{p=0} = \\ = \left\{\frac{\exp(pt)}{2}\left[t^2 L(p) + 2t\dot{L}(p)\right] + \ddot{L}(p)\right\}_{p=0} \end{gathered}. \tag{28}$$

The functions $f_{(IV)}$ and its' time derivative $\dot{f}_{(IV)}$ have look like

$$f_{(IV)} = \frac{1}{2}\left(t^2 + 2ta_1 + 2a_2\right), \tag{29}$$

$$\dot{f}_{(IV)} = t + a_1. \tag{30}$$

Assuming (14.1-2) and (30) we arrive at

$$b_2\varepsilon_{(IV)} = \sigma_0\left[\frac{t^2 - t_0{}^2}{2} + (t-t_0)a_1\right] \rightarrow \sigma_0\frac{t^2 - t_0{}^2}{2} \tag{31.1}$$

for $t_0 \le t \le t_1$ and

$$b_2\varepsilon_{(IV)} = \sigma_0\frac{t-t_0}{2}\left[2t - (t_1 - t_0) + 2a_1\right] \rightarrow \frac{\sigma_0}{2}\left[2t^2 - t(t_0 + t_1) + t_0(t_1 - t_0)\right] \tag{31.2}$$

since $t > t_1$.

4. The case of Burgers model

Treating the above results as a generalization we can proceed to particular models, here – to the Burgers model. In the case of (I), setting up –

$$\beta_0 = 0 \tag{32}$$

we obtain

$$p_1 = 0 \text{ and } p_2 = -\beta_1 = -\frac{b_1}{b_2}. \tag{33}$$

Again, we have a double singularity point, now for

$$p_0 = p_1 = 0, \tag{34}$$

the second one, non zero, is p_2. The values of residua are

$$p_0 = p_1 = 0 \to \left\{\frac{d}{dp}\frac{[\exp(pt)L(p)]}{p+\beta_1}\right\}_{p=0} = \\ = \left\{\frac{\exp(pt)\{[t\,L(p)+\dot{L}(p)](p+\beta_1)-L(p)\}}{(p+\beta_1)^2}\right\}_{p=0} \tag{35.1}$$

$$p_2 = -\beta_1 \to \left\{\frac{\exp(pt)L(p)}{p^2}\right\}_{p=-\beta_1}. \tag{35.2}$$

The above results in

$$f_{(B.)} = \frac{1}{(\beta_1)^2}\left[(t+a_1)\beta_1 - 1 + \exp(-\beta_1 t)L(-\beta_1)\right] \text{ and} \tag{36}$$

$$\dot{f}_{(B.)} = \frac{1}{\beta_1}\left[1-\exp(-\beta_1 t)L(-\beta_1)\right]. \tag{37}$$

Considering the load function (14.1) we arrive at

$$b_2\varepsilon = \frac{\sigma_0}{(\beta_1)^2}\{(t-t_0)\beta_1 - [1-\exp(-\beta_1(t-t_0))]L(-\beta_1)\} \tag{38.1}$$

when $t_0 \le t \le t_1$ and

$$b_2\varepsilon_{(B.)} = \frac{\sigma_0}{(\beta_1)^2}\left[-\beta_1(t_1-t_0) + \mathrm{Exp}(-\beta_1 t)(\exp(\beta_1 t_1) - \exp(\beta_1 t_0))L(-\beta_1)\right] \tag{38.2}$$

if $t > t_1$; now, taking into account (14.2) we get the Burgers model

$$\varepsilon_{(B.)} = \sigma_0 \frac{b_2}{(b_1)^2}\left[\frac{b_1}{b_2}(t-t_0) - 1 + \exp\left(-\frac{b_1}{b_2}(t-t_0)\right)\right], \qquad t_0 \le t \le t_1; \qquad (39.1)$$

$$\varepsilon_{(B.)} = -\sigma_0 \frac{b_2}{(b_1)^2}\left\{\frac{b_1}{b_2}(t_1-t_0) + \exp\left(-\frac{b_1}{b_2}t\right)\left[\exp\left(\frac{b_1}{b_2}t_1\right) - \exp\left(\frac{b_1}{b_2}t_0\right)\right]\right\}, \qquad (39.2)$$

for $t > t_1$.

7. **References**

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Rozwiązanie kompletne uogólnionego modelu Kelvina-Voigta

Użyteczność jednowymiarowych modeli lepkosprężystych, szczególnie w zagadnieniach nawierzchni drogowych, kompozytach i innych dziedzinach inżynierii lądowej stała się przyczyną podjęcia próby znalezienia kompletnego rozwiązania uogólnionego modelu Kelvina-Voigta, przy czym w modelu uwzględniono także przyspieszenia tak naprężeń jak i odkształceń. Do uzyskania rozwiązań wykorzystano transformację Carsona oraz twierdzenie o residuach. Zastosowana procedura może być także użyta w przypadkach bardziej złożonych związków konstytutywnych w formie różniczkowej lub całkowej, jak również przy niejednorodnych warunkach początkowych. Na zakończenie rozpatrzono szczególny przypadek analizowanego uogólnienia tj. model Burgersa.